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TORSION CONSTANT FOR MATRIX ANALYSIS OF STRUCTURES INCLUDING WARPING EFFECT

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Abstract—In this research, an effective torsion constant, J_{eff}, for a wide-flanged member with different warping restraint conditions at the ends is developed. In order to derive J_{eff}, the boundary conditions for different warping restraint conditions at the ends, as imposed by the use of different types of connections in steel structures, are employed in the general equation of torsional rotation, ϕ. This effective torsion constant can be used directly instead of the St Venant torsion constant, J, in the conventional member stiffness matrix. The use of J_{eff} will account for the effect of warping when using the commercial computer programs which employ either a 6×6 member stiffness matrix for grid or a 12×12 member stiffness matrix for space frame. A table for factor F, which is the ratio of J_{eff} to J, is presented for different member properties and warping restraint conditions for the easy and rapid torsional analysis of structures composed of wide-flanged members by the matrix method. The solutions of a sample grid problem using J_{eff} are compared with well-known solutions to demonstrate its application.

NOTATION

A_1, A_2, A_3	constants of integration
a, b	ends of a member
B	bimoment
b_s	effective distance between springs
C_w	warping constant
d	effective depth of member
E	modulus of elasticity
e	deformation of linear spring
F	a factor, which is the ratio of J_{eff} to J
F_s	linear spring resistance
$[f]$	member flexibility matrix
G	shear modulus
I_x, I_y	moment of inertia
J	St Venant torsion constant
J_{eff}	effective torsion constant
k_s	linear spring constant
$[k]$	stiffness matrix of a member for torsional loading
L	length of a member
M_f	lateral bending moment of the flange
p	a constant equal to $\sqrt{GJ/EC_w}$
s	warping spring constant
s_a, s_b	warping spring constants at ends a, b
$\bar{s}_a, \bar{s}_b$	dimensionless warping spring constants at ends a, b
T	torsional moment
T_s	St Venant torsion
T_w	warping torsion
w	warping indicator
w_a, w_b	warping indicator at ends a, b
ϕ	angle of twist
$\phi' = (d\phi/dx)_{x=0}$	rate of twist
θ	in-plane rotation of flange

INTRODUCTION

In steel structures, the grid and space frame may experience significant torsion in the members due to applied loads and even self weight. Accurate analysis of such systems is difficult to achieve using available commercial programs because the torsional stiffness associated with warping restraint at the ends cannot be included in these programs (generally the available programs consider three degrees of freedom at each node for a grid and six degrees of freedom at each node for a space frame); but, if such structures are analyzed or designed for torsion considering only the effect of St Venant torsion resistance, the analysis may underestimate the torsion in the members and the design may then be unconservative.

When a member with wide-flanged cross-section is subjected to torsional loading, the member experiences large out-of-plane warping displacements at the ends. If these warping displacements are fully or partially restrained by some means (elements of connections), a system of torsion and bimoment is produced. In that case, the applied torsional load is resisted by two components, the St Venant torsion, T_s, and the warping torsion, T_w. Since the out-of-plane warping displacements are relatively large in steel structures where the members are made of wide-flanged cross-sections, the effect of warping resistance is significant. Thus analysis which includes the effect of warping restraint is desired in the structural grid and space frame.

In early investigations, several researchers assumed that the member ends were either free to warp or completely prevented from warping. Reilly (1972), for example, assumed warping is fully restrained and developed an 8×8 element stiffness matrix including the warping effect for the displacement method of analysis for the structural grid. He considered the warping effect, measured by the rate of twist, $d\phi/dx$, as an additional degree of freedom at each end of a grid element. In order to include the effect of warping in space frame, Barsoum and Gallegher (1970) developed a 14×14 member stiffness matrix including warping degree of freedom. In the recent past, Waldron (1985, 1986) derived a member stiffness matrix for thin-walled girders. This member stiffness matrix is derived explicitly by inverting the appropriate member flexibility matrix and considering the equilibrium of the member. It is noted that the stiffness matrix for an open, thin-walled section is identical to one of the matrices developed by Reilly (1972).

In reality, the effect of warping restraint on the torsional stiffness depends upon the amount of resistance that can be mobilized at the member ends. The warping resistance is a function of the cross-sectional dimensions, the length of the member and the joint details at the ends of the member, i.e. the type of connections. Thus, in steel structures, a partial warping restraint condition may arise at the ends depending upon the joint details. Considering this fact, Yang and McGuire (1984) introduced a procedure for analyzing space frame with partial warping restraint at the ends. They introduced the elastic warping spring concept and the term warping indicator to represent the partial warping restraint for each end of the member. The warping indicator is defined as the ratio of the warping deformation at the end of a member when that end is partially restrained to the warping deformation at the same end when it is free to warp. Using this concept, Yang and McGuire (1984) applied a static condensation procedure to one of the available member stiffness matrices including warping degree of freedom, and thus eliminated the non-continuous warping degrees of freedom associated with the restrained warping (a non-continuous end is defined as the one at which the value of warping is not the same as that at the adjoining end of the next member).

The problem with using the stiffness matrices which include additional degrees of freedom for warping, is that all the available commercial computer programs for the analysis of structures employ either a 6×6 element stiffness matrix for a grid or a 12×12 element stiffness matrix for a space frame. The static condensation procedure can be employed in the torsional analysis of members including warping, but again the modification of the ordinary member stiffness matrices including warping is required according to the degree of partial restraint at the ends of the member.

This research work develops an effective torsion constant, J_{eff}, for wide-flanged members which accounts for the effect of warping restraints at the ends resulting from different

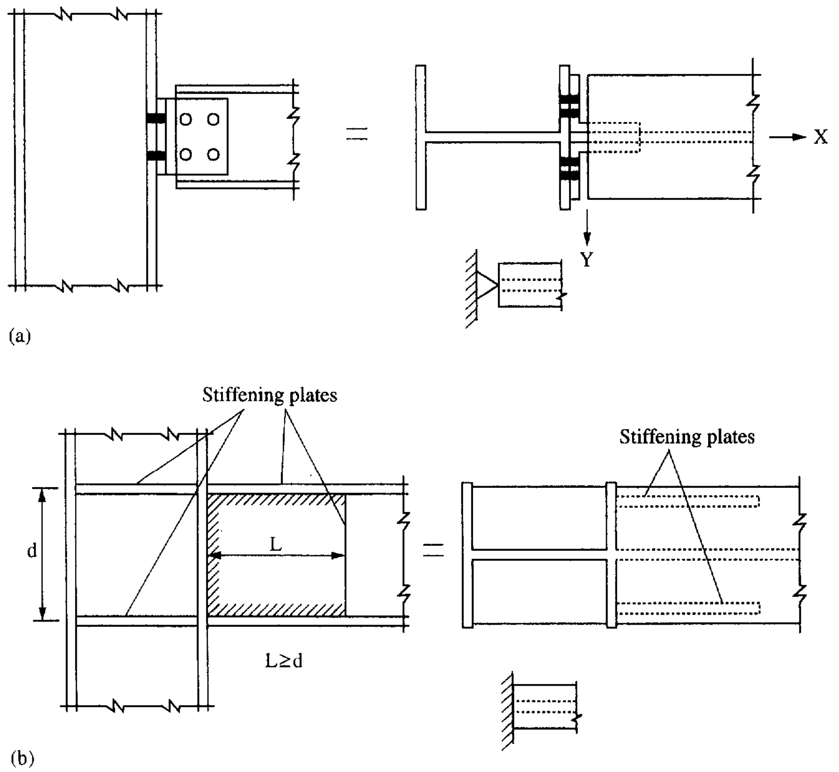

Fig. 1. Torsional end restraint conditions. (a) Simple framing; (b) rigid framing connections [adapted from Salmon and Johnson (1980)].

categories of construction as permitted by AISC. The concept of warping spring stiffness and warping indicator (Yang and McGuire, 1984) is employed to derive J_{eff} for partial warping restraint conditions. This J_{eff} can be used directly instead of the St Venant torsion constant, J, in the available commercial programs.

TORSIONAL END RESTRAINTS

According to AISC 1.2 and 2.1, three categories of construction are permitted in steel structures: type I—"rigid frame", i.e. full flexural restraint at joints; type II—"simple", i.e. negligible flexural restraint at joints; type III—"semi-rigid framing" which is in-between type I and type II constructions. Example connections for type I and II constructions are illustrated in Fig. 1. The type I connection shown in Fig. 1(b) includes stiffeners to enhance warping restraint. In the type II connection shown in Fig. 1(a), web angles are used to connect the web of the beam to the flange of the column and the angles are designed to be as flexible as possible. The lateral bending analogy can be used in describing the torsional restraint conditions. Figure 1(a) shows the flanges at the member ends (type II connection) to have zero deflection and zero moment which correspond torsionally to $\phi = 0$ and $d^2\phi/dx^2 = 0$. Figure 1(b) shows the analogous situation of zero deflection and zero slope which correspond torsionally to $\phi = 0$ and $d\phi/dx = 0$ (type I connections). Similarly, the boundary conditions for type III connections are $\phi = 0$ and bimoment, $B = EC_w(d^2\phi/dx^2)$. These boundary conditions are applied to derive J_{eff} as given in the following.

EFFECTIVE TORSION CONSTANT, J_{eff}

Case I: *member with torsionally rigid framing* (*type* I) *connections at both ends*

Figure 2(a) shows a wide-flanged member which is free to twist at end b and prevented from twisting at end a. The out-of-plane displacements at both ends are fully prevented. A

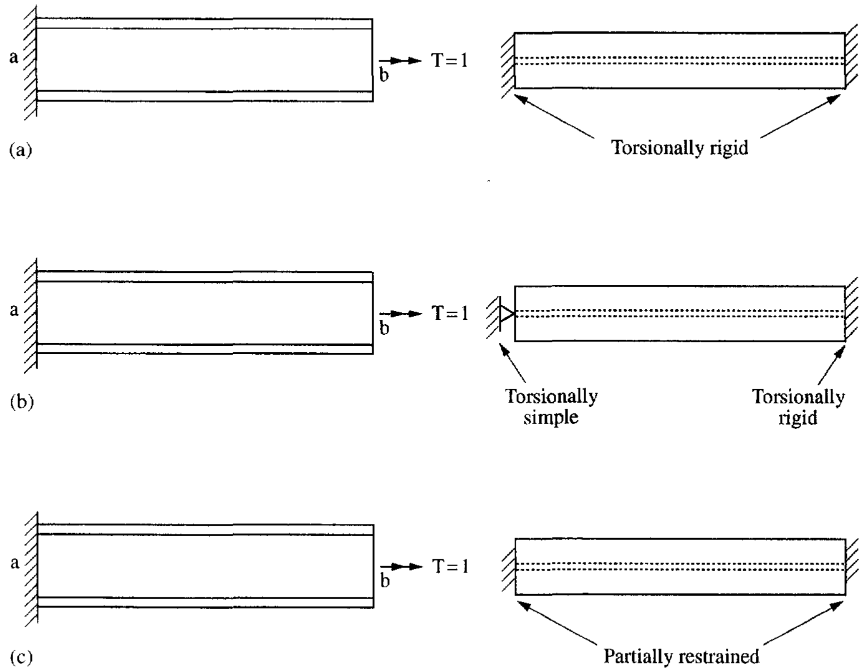

Fig. 2. Member with different types of end restraints. (a) Torsionally rigid connection at both ends; (b) torsionally simple connection at end a and torsionally rigid connection at end b; (c) partial warping restraint at both ends.

torque T is applied at end b. This applied torque T is resisted internally by two components, the St Venant torsion T_s and the warping torsion T_w, which results from the restraint of out-of-plane displacements. The St Venant torsion, warping torsion and bimoment, B, are related to torsional rotation, ϕ, as:

$$T_s = GJ\frac{d\phi}{dx} \tag{1}$$

$$T_w = -EC_w\frac{d^3\phi}{dx^3} \tag{2}$$

$$B = -EC_w\frac{d^2\phi}{dx^2}. \tag{3}$$

Thus the total torsion, T, can be expressed as:

$$\frac{d^3\phi}{dx^3} - p^2\frac{d\phi}{dx} = -\frac{T}{EC_w}, \tag{4}$$

in which

$$p^2 = \frac{GJ}{EC_w}. \tag{5}$$

The general solution of eqn (4) is:

$$\phi = \frac{Tx}{GJ} + A_1 \sinh(px) + A_2 \cosh(px) + A_3, \tag{6}$$

Applying the following boundary conditions in eqn (6),

$$\phi = 0 \text{ at } x = 0$$

$$\frac{d\phi}{dx} = 0 \text{ at } x = 0$$

$$\frac{d\phi}{dx} = 0 \text{ at } x = L,$$

will give:

$$0 = A_2 + A_3 \tag{7}$$

$$0 = \frac{T}{GJ} + A_1 p \tag{8}$$

and

$$0 = \frac{T}{GJ} + A_1 p \cosh(pL) + A_2 p \sinh(pL). \tag{9}$$

On solving eqns (7)–(9), the constants of integration can be obtained as

$$A_1 = -\frac{T}{GJp} \tag{10}$$

and

$$A_2 = -A_3 = \frac{T}{GJ} \cdot \frac{\{\cosh(pL) - 1\}}{p \sinh(pL)}. \tag{11}$$

Substitution of these constants in eqn (6) leads to the following equation for angle of twist, ϕ:

$$\phi = \frac{Tx}{GJ} - \frac{T}{GJp} \cdot \sinh(px) + \frac{T\{\cosh(pL) - 1\}}{p \sinh(pL)} \cdot \cosh(px) - \frac{T\{\cosh(pL) - 1\}}{p \sinh(pL)}. \tag{12}$$

Assuming the applied torque, T, to be unity at end b, the twist angle becomes:

$$\phi_{x=L} = f_{bb} = \frac{L}{G} \cdot (1 - a_1 + a_2) \tag{13}$$

and the inversion of f_{bb} gives the stiffness coefficient k_{bb} as:

$$k_{bb} = [\mathbf{f}_{bb}]^{-1} = \frac{GJ_{eff}}{L}, \tag{14}$$

where

$$J_{\text{eff}} = J\left(\frac{1}{1-a_1+a_2}\right) \tag{15}$$

$$a_1 = \frac{\sinh(pL)}{pL} \tag{16}$$

and

$$a_2 = \frac{\{\cosh(pL)-1\}^2}{pL\sinh(pL)}. \tag{17}$$

Thus the stiffness matrix of the member for torsional loading is:

$$[\mathbf{k}] = \frac{GJ_{\text{eff}}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{18}$$

Case II : *member with one end a simple connection and the other end a rigid framing connection*

The member shown in Fig. 2(b) has the end connections which allow warping at end a and prevent warping at end b. A torsional load, T, is applied to end b as before. The boundary conditions for this member are as follows:

$$\phi = 0 \text{ at } x = 0$$

$$\frac{d^2\phi}{dx^2} = 0 \text{ at } x = 0$$

$$\frac{d\phi}{dx} = 0 \text{ at } x = L.$$

Upon substitution of these boundary conditions for this member and on application of the same technique as in case I, the stiffness coefficient, k_{bb}, is obtained as:

$$k_{bb} = [\mathbf{f}_{bb}]^{-1} = \frac{GJ_{\text{eff}}}{L}, \tag{19}$$

where

$$J_{\text{eff}} = J\left(\frac{1}{1-a_3}\right) \tag{20}$$

and

$$a_3 = \frac{\sinh(pL)}{pL\cosh(pL)}. \tag{21}$$

Case III : *member with partial warping restraint at both ends*

The J_{eff} for a member with partial warping restraint conditions as shown in Fig. 2(c) is derived by employing the warping spring concept and the term warping indicator as described by Yang and McGuire (1984). As mentioned earlier, the term warping indicator is the ratio of the warping deformation at one end of a member when that end is partially restrained to the warping deformation at the same end when it is free to warp. This warping indicator varies from zero when there is no warping to unity when there is no resistance to warping. For this case, i.e. the member with warping support at the ends (elastic warping support or spring is imposed at the end of the member to model the partial warping restraint conditions as shown in Fig. A1), the bimoment, B, at the end of the member is (Appendix):

$$B = s\frac{d\phi}{dx}, \tag{22}$$

where s is the warping spring constant. The boundary conditions required to find the constants of integration of the general equation (6) are:

$$\phi = 0 \text{ at } x = 0 \tag{23}$$

$$B = -EC_w\frac{d^2\phi}{dx^2} = -s_a\frac{d\phi}{dx} \text{ at } x = 0$$

or

$$\frac{d^2\phi}{dx^2} = \frac{s_a}{EC_w}\cdot\frac{d\phi}{dx} \text{ at } x = 0 \tag{24}$$

and

$$B = EC_w\frac{d^2\phi}{dx^2} = -s_b\frac{d\phi}{dx} \text{ at } x = L$$

or

$$\frac{d^2\phi}{dx^2} = -\frac{s_b}{EC_w}\cdot\frac{d\phi}{dx} \text{ at } x = L, \tag{25}$$

The difference in sign between eqns (24) and (25) is required by the equilibrium of bimoment at the right-hand side of the member. Applying these boundary conditions to eqn (6), the constants of integration are found to be:

$$A_1 = -\frac{T}{GJ}\cdot\frac{s_a + s_b\cosh(pL) + \frac{s_a s_b}{GJ}p\sinh(pL)}{p(s_a+s_b)\cosh(pL) + \left(GJ + \frac{s_a s_b}{EC_w}\right)\sinh(pL)} \tag{26}$$

$$A_2 = -\frac{T}{GJ}\cdot\frac{s_a\sinh(pL) - \frac{s_a s_b}{GJ}p\{1-\cosh(pL)\}}{p(s_a+s_b)\cosh(pL) + \left(GJ + \frac{s_a s_b}{EC_w}\right)\sinh(pL)} \tag{27}$$

and

$$A_3 = -A_2. \tag{28}$$

Substituting these constants in eqn (6) and on applying the same procedure as described earlier, J_{eff} for the member with partial warping restraint condition is obtained as:

$$J_{\text{eff}} = J\left(\frac{1}{1-B_1[\sinh(pL)/pL]+B_2[\cosh(pL)-1/pL]}\right), \tag{29}$$

where

$$B_1 = \frac{\hat{s}_b+\hat{s}_a\cosh(pL)+\hat{s}_a\hat{s}_b\sinh(pL)}{(\hat{s}_a+\hat{s}_b)\cosh(pL)+(1+\hat{s}_a\hat{s}_b)\sinh(pL)} \tag{30}$$

and

$$B_2 = \frac{\hat{s}_a\sinh(pL)-\hat{s}_a\hat{s}_b\{1-\cosh(pL)\}}{(\hat{s}_a+\hat{s}_b)\cosh(pL)+(1+\hat{s}_a\hat{s}_b)\sinh(pL)}. \tag{31}$$

In the above expressions, $\hat{s}_a$ and $\hat{s}_b$ are the non-dimensional terms as given in the following:

$$\hat{s}_a = \frac{s_a}{(GJEC_w)^{1/2}} \tag{32}$$

$$\hat{s}_b = \frac{s_b}{(GJEC_w)^{1/2}}. \tag{33}$$

Thus the J_{eff} for a member with partial warping restraint at the ends depends upon the non-dimensional terms $\hat{s}_a$ and $\hat{s}_b$ which can be calculated by solving the following equations as derived by Yang and McGuire (1984):

$$\hat{s}_a w_a+\hat{s}_b(w_b-1)+\hat{s}_a\hat{s}_b w_a\tanh(pL) = (1-w_a)\tanh(pL) \tag{34}$$

and

$$\hat{s}_a(w_b-1)+\hat{s}_b w_b+\hat{s}_a\hat{s}_b w_b\tanh(pL) = (1-w_b)\tanh(pL). \tag{35}$$

It can be noted that the coefficients of eqns (34) and (35) are functions of the warping indicators, w_a and w_b, which vary from zero as s tends to infinity (no warping at the end) to unity as s tends to zero (no restraining bimoment).

Thus the torsion constant of a member including the effect of warping is equal to the St Venant torsion constant, J, times a factor, F. This factor F can be determined for different member properties pL and warping boundary conditions at the member ends. In other words, the factor F can be determined if the values of the warping indicator w at the ends of a member and the parameter pL are known. Since the warping indicator is based on the physical deformation, it can be measured or estimated indirectly by judging the joint details.

In order to facilitate the use of commercial packages, the factor F is calculated for different member properties pL and warping indicators w_a and w_b. This factor F (Table 1) can be easily employed in the matrix method of structural analysis including the warping effect when using the available commercial packages.

SAMPLE SOLUTION

A simple grid structure composed of wide-flanged members with various warping boundary conditions [Fig. 3(a)] is used as a sample problem for the matrix analysis including the warping effect. This grid is the same as that used by Yang and McGuire (1984). The STRUDL-II package for the stiffness analysis of the frame structures is employed to analyze the sample grid. The effective torsion constant, J_{eff}, is used in the analyses as the requisite input torsional constant in lieu of simply using J, i.e. the St Venant torsion constant. All the members of the grid are W 36 × 230 beams (Yang and McGuire, 1984). The lengths of the members AB and BC are 60 and 600 in, respectively. The properties are: $I_z = 15{,}000$ in^4, $I_y = 940$ in^4, $J = 28.6$ in^4, $C_w = 282{,}000$ in^6 (I is moment of inertia). The modulus of elasticity and rigidity are $E = 29{,}000$ ksi and $G = 11{,}200$ ksi, respectively. The member properties, $pL = \sqrt{(GJ/EC_w)}L$, are 0.376 and 3.751 for members AB and BC, respectively. A torsional moment of 100 kip-in is applied at the corner of the grid as shown in Fig. 3(a). In the first example, this grid is analyzed considering the ends of the members fully restrained against warping (torsionally rigid, i.e. type I connections and the warping indicators w_a, w_b and w_c are zero) and the corresponding J_{eff} are 2512.8 and 57.5 in^4 for members AB and BC, respectively (the factor F is calculated as 87.86 and 2.01 for members AB and BC, respectively, from Table 1). Results of this analysis, i.e. the torsion and the moments, are summarized in Fig. 3. The second example considers the same grid but with different warping boundary conditions. The supports A and C are considered to be torsionally rigid (type I connections; w_a and w_c are zero). The ends of the members AB and BC at the corner joint B are considered to be partially restrained against warping with a value of warping indicator $w_b = 0.5$. For comparison, the result of this analysis is shown in Fig. 3 (in parentheses). It can be noted that distribution of torsion, T, in the members is greatly dependent upon the warping restraints.

The grid is then analyzed using the St Venant torsion constant, J. A comparison of torsion as obtained in Example 1 using J_{eff} (the warping indicators w_a, w_b and w_c are zero)

Table 1. Factor F for different member properties pL and warping indicator w

pL	$w_a = 0.0$ $w_b = 0.0$	$w_a = 0.0$ $w_b = 0.25$	$w_a = 0.0$ $w_b = 0.50$	$w_a = 0.0$ $w_b = 0.75$	$w_a = 0.0$ $w_b = 1.0$	$w_a = 0.25$ $w_b = 0.25$	$w_a = 0.25$ $w_b = 0.50$	$w_a = 0.25$ $w_b = 0.75$	$w_a = 0.25$ $w_b = 1.0$
0.1	1201.1	687.5	481.6	370.6	300.3	303.4	135.0	24.0	4.0
0.2	301.2	173.3	121.6	93.7	76.1	78.4	37.3	12.4	3.8
0.3	134.5	78.0	55.0	42.4	34.5	36.6	18.9	8.5	3.7
0.4	76.2	44.7	31.6	24.5	19.9	21.9	12.1	6.5	3.5
0.5	49.2	29.3	20.8	16.2	13.2	15.0	8.9	5.4	3.3
0.6	34.5	20.9	14.9	11.6	9.5	11.2	7.0	4.6	3.0
0.7	25.7	15.8	11.4	8.9	7.3	8.9	5.8	4.0	2.8
0.8	19.9	12.5	9.1	7.1	5.9	7.3	4.9	3.6	2.6
0.9	16.0	10.2	7.5	5.9	4.9	6.2	4.3	3.2	2.5
1.0	13.2	8.6	6.4	5.1	4.2	5.4	3.9	3.0	2.3
1.1	11.1	7.4	5.5	4.4	3.7	4.8	3.5	2.7	2.2
1.2	9.5	6.5	4.9	3.9	3.3	4.3	3.2	2.6	2.1
1.3	8.3	5.7	4.4	3.5	3.0	4.0	3.0	2.4	2.0
1.4	7.3	5.1	4.0	3.2	2.7	3.7	2.8	2.3	1.9
1.5	6.5	4.7	3.6	3.0	2.5	3.4	2.7	2.2	1.8
1.6	5.9	4.3	3.4	2.8	2.4	3.2	2.5	2.1	1.8
1.7	5.3	4.0	3.1	2.6	2.2	3.0	2.4	2.0	1.7
1.8	4.9	3.7	2.9	2.5	2.1	2.8	2.3	1.9	1.7
1.9	4.5	3.4	2.8	2.3	2.0	2.7	2.2	1.9	1.6
2.0	4.2	3.2	2.6	2.2	1.9	2.6	2.1	1.8	1.6
2.1	3.9	3.1	2.5	2.1	1.9	2.5	2.0	1.8	1.5
2.2	3.7	2.9	2.4	2.1	1.8	2.4	2.0	1.7	1.5
2.3	3.5	2.8	2.3	2.0	1.7	2.3	1.9	1.7	1.5
2.4	3.3	2.7	2.2	1.9	1.7	2.2	1.9	1.6	1.4
2.5	3.1	2.5	2.2	1.9	1.7	2.1	1.8	1.6	1.4
2.6	3.0	2.5	2.1	1.8	1.6	2.1	1.8	1.6	1.4
2.7	2.8	2.4	2.0	1.8	1.6	2.0	1.7	1.5	1.4
2.8	2.7	2.3	2.0	1.7	1.5	2.0	1.7	1.5	1.4
2.9	2.6	2.2	1.9	1.7	1.5	1.9	1.7	1.5	1.3
3.0	2.5	2.2	1.9	1.7	1.5	1.9	1.6	1.5	1.3
3.1	2.4	2.1	1.8	1.6	1.5	1.8	1.6	1.5	1.3
3.2	2.4	2.0	1.8	1.6	1.5	1.8	1.6	1.4	1.3
3.3	2.3	2.0	1.8	1.6	1.4	1.8	1.6	1.4	1.3
3.4	2.2	1.9	1.7	1.6	1.4	1.7	1.5	1.4	1.3
3.5	2.2	1.9	1.7	1.5	1.4	1.7	1.5	1.4	1.3
3.6	2.1	1.9	1.7	1.5	1.4	1.7	1.5	1.4	1.3
3.7	2.1	1.8	1.6	1.5	1.4	1.6	1.5	1.4	1.3
3.8	2.0	1.8	1.6	1.5	1.4	1.6	1.5	1.3	1.2
3.9	2.0	1.8	1.6	1.5	1.3	1.6	1.5	1.3	1.2
4.0	1.9	1.7	1.6	1.4	1.3	1.6	1.4	1.3	1.2

(*Continued overleaf*)

Table 1—*Continued*

pL	$w_a = 0.5$ $w_b = 0.5$	$w_a = 0.5$ $w_b = 0.75$	$w_a = 0.5$ $w_b = 1.0$	$w_a = 0.75$ $w_b = 0.75$	$w_a = 0.75$ $w_b = 1.0$	$w_a = 1.0$ $w_b = 1.0$
0.1	20.7	3.9	2.0	2.0	1.3	1.0
0.2	10.7	3.6	2.0	2.0	1.3	1.0
0.3	7.3	3.3	1.9	1.9	1.3	1.0
0.4	5.7	3.0	1.9	1.9	1.3	1.0
0.5	4.7	2.8	1.9	1.8	1.3	1.0
0.6	4.0	2.6	1.8	1.7	1.3	1.0
0.7	3.6	2.4	1.8	1.7	1.3	1.0
0.8	3.2	2.3	1.7	1.6	1.3	1.0
0.9	2.9	2.2	1.7	1.6	1.2	1.0
1.0	2.7	2.0	1.6	1.5	1.2	1.0
1.1	2.5	2.0	1.6	1.5	1.2	1.0
1.2	2.4	1.9	1.5	1.5	1.2	1.0
1.3	2.3	1.8	1.5	1.4	1.2	1.0
1.4	2.2	1.8	1.5	1.4	1.2	1.0
1.5	2.1	1.7	1.4	1.4	1.2	1.0
1.6	2.0	1.7	1.4	1.4	1.2	1.0
1.7	1.9	1.6	1.4	1.3	1.2	1.0
1.8	1.9	1.6	1.4	1.3	1.2	1.0
1.9	1.8	1.5	1.3	1.3	1.1	1.0
2.0	1.8	1.5	1.3	1.3	1.1	1.0
2.1	1.7	1.5	1.3	1.3	1.1	1.0
2.2	1.7	1.5	1.3	1.3	1.1	1.0
2.3	1.6	1.4	1.3	1.3	1.1	1.0
2.4	1.6	1.4	1.3	1.2	1.1	1.0
2.5	1.6	1.4	1.2	1.2	1.1	1.0
2.6	1.6	1.4	1.2	1.2	1.1	1.0
2.7	1.5	1.4	1.2	1.2	1.1	1.0
2.8	1.5	1.3	1.2	1.2	1.1	1.0
2.9	1.5	1.3	1.2	1.2	1.1	1.0
3.0	1.5	1.3	1.2	1.2	1.1	1.0
3.1	1.4	1.3	1.2	1.2	1.1	1.0
3.2	1.4	1.3	1.2	1.2	1.1	1.0
3.3	1.4	1.3	1.2	1.2	1.1	1.0
3.4	1.4	1.3	1.2	1.2	1.1	1.0
3.5	1.4	1.3	1.2	1.2	1.1	1.0
3.6	1.4	1.3	1.2	1.2	1.1	1.0
3.7	1.4	1.2	1.2	1.2	1.1	1.0
3.8	1.3	1.2	1.2	1.1	1.1	1.0
3.9	1.3	1.2	1.1	1.1	1.1	1.0
4.0	1.3	1.2	1.1	1.1	1.1	1.0

with those using J only is presented in Fig. 4 (the result using J is shown in parentheses). It can be observed that the torsion in the members increases significantly (more than 100%) due to the effect of warping restraints at the joints. Thus the comparison suggests that if the resistance against warping is large, the effect of warping restraints must be considered in the analysis.

To determine how the use of J_{eff} compares with other available analyses, the results as obtained in the first example using J_{eff} [bimoment is calculated applying the principle of flexural analogy; Salmon and Johnson (1980)] are compared with those obtained by using the member stiffness matrix (8×8) including warping degree of freedom as developed by Reilly (1972). Figure 5 shows that the result obtained using J_{eff} agrees well with those obtained using the stiffness matrix including the warping degree of freedom. This comparison demonstrates that the effect of warping can be incorporated in the matrix analysis by employing J_{eff} in the conventional member stiffness matrix. In other words, the available commercial packages can be employed in determining the actual torsion in the members of a grid or a space frame if J_{eff} is used as input data instead of J. It can also be mentioned here that although the use of J_{eff} does not provide bimoment, which is needed to calculate flange moment, M_f, due to warping torsion, this can be estimated conservatively for design purposes by applying the technique of flexural analogy (Salmon and Johnson, 1980).

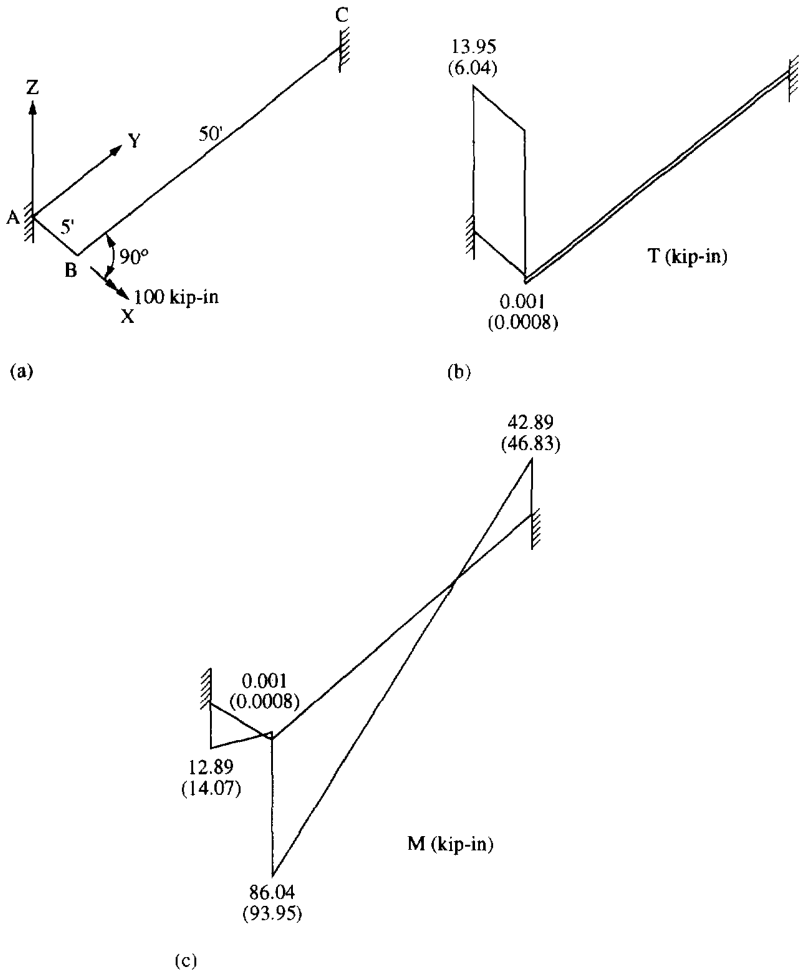

Fig. 3. Sample grid with a concentrated torsional loading. (a) Geometry; (b) torque diagram; (c) moment diagram.

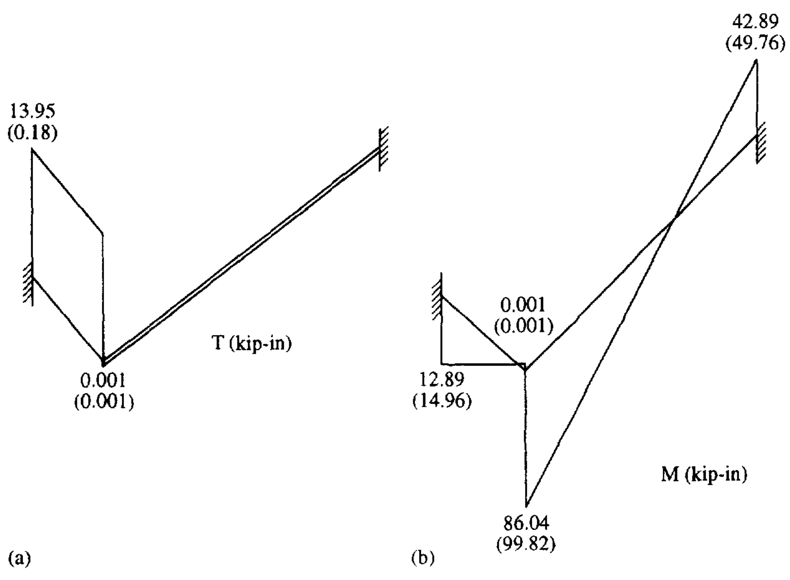

Fig. 4. A comparison of results as obtained using J_{eff} with those using J.

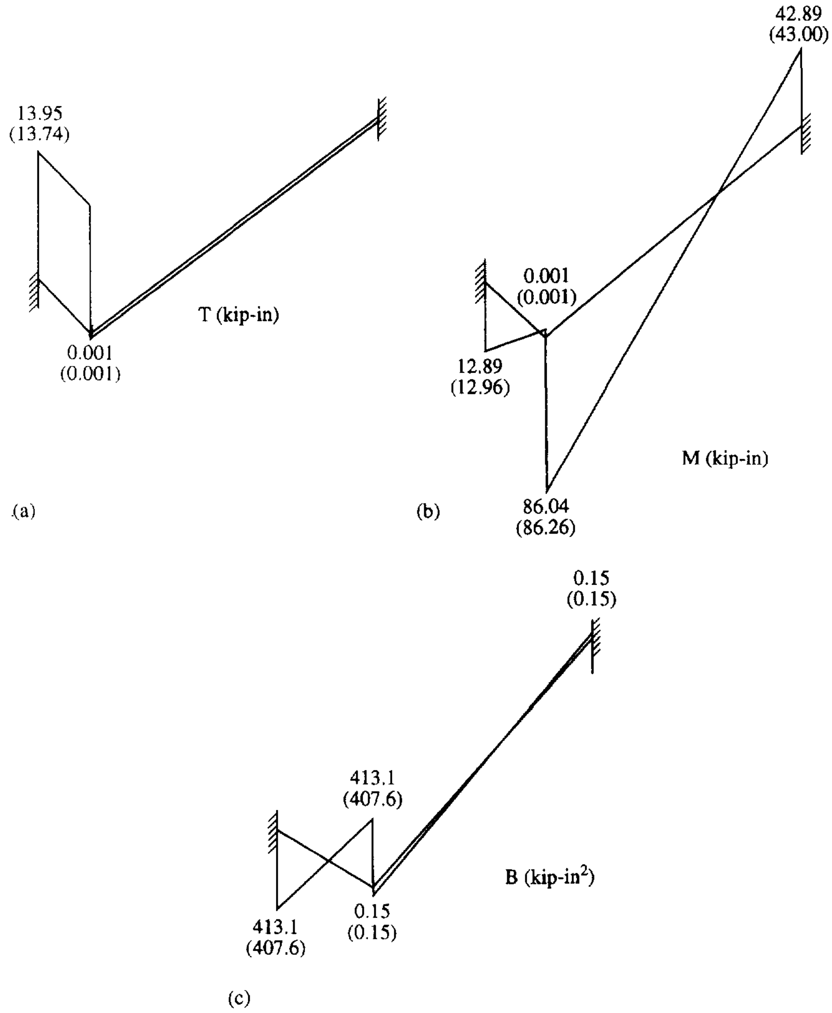

Fig. 5. A comparison of results as obtained using J_{eff} with those using member stiffness matrix including warping degree of freedom.

CONCLUSIONS

The analysis of the grid and space frame composed of wide-flanged members (steel structures) is underestimated if they are analyzed by the stiffness method using only the St Venant torsion constant, J. It is recommended in this research that a solution of these structures including the effect of warping can be obtained by the stiffness method, using J_{eff} instead of J in the member stiffness matrix. Since most of the available commercial programs employ either a 6×6 member stiffness for grids or a 12×12 member stiffness matrix for the space frame, the use of the effective torsion constant J_{eff} as derived in this research for different boundary conditions (member with type I and type II connections) can be easy and convenient to account for the warping effect when using these commercial packages. Moreover, this J_{eff} can be easily employed for the analysis of structures with partial warping restraints (type III connections) provided the amount of warping restraint at the ends is estimated.

Several analyses of a sample structural grid with various warping boundary conditions are carried out by the stiffness matrix with J_{eff} and J. From these analyses the following conclusions can be drawn.

The exact distribution of torsion, T, in the members is dependent upon the warping boundary conditions, i.e. the degrees of warping restraint at the joints.

The analysis of steel structures gives conservative results when using the effective torsion constant, J_{eff}, instead of the St Venant torsion constant, J, in the conventional member stiffness matrix.

Excellent agreement is obtained between the results as obtained using J_{eff} in the conventional member stiffness matrix with those obtained using the member stiffness matrix including the warping degree of freedom.

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APPENDIX: WARPING SPRING CONSTANT

As given by Yang and McGuire (1984), Fig. A1 illustrates the warping spring concept schematically, where linear elastic restraints act on the end of a wide-flanged member. The symbols used in the figure are defined as: k_s, linear spring constant (kips per inch); d, effective depth of member (inch) b_s, effective distance between springs; $\phi' = (d\phi/dx)_{x=0}$ = rate of twist at end a, (radians per inch); θ, in-plane rotation of flange; e, deformation of linear spring; and F_s, linear spring resistance.

Point a is the reference point where the twist is taken as zero for the warped configuration of end of a differential element of length dx. The stiffness of the warping spring (warping spring constant) can be obtained from:

$$\theta = \frac{(d/2)\,d\phi}{dx} = \frac{d}{2}\frac{d\phi}{dx} \tag{A1}$$

$$e = \frac{b_s}{2}\theta = \frac{b_s d}{4}\frac{d\phi}{dx} \tag{A2}$$

$$F_s = k_s e = \frac{k_s b_s d}{4}\frac{d\phi}{dx} \tag{A3}$$

$$M_f = F_s b_s = \frac{k_s b_s^2 d}{4}\frac{d\phi}{dx}, \quad \text{lateral bending moment of flange} \tag{A4}$$

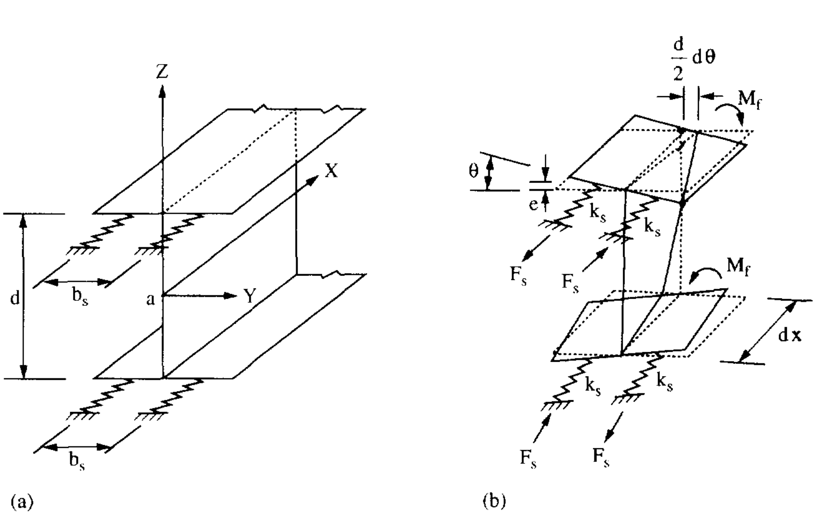

Fig. A1. Physical model for warping spring. (a) Undeformed configuration; (b) warped configuration [adapted from Yang and McGuire (1984)].

$$B = M_f d = \frac{k_s b_s^2 d^2}{4} \frac{d\phi}{dx}, \quad \text{bimoment}$$

or

$$B = s\left(\frac{d\phi}{dx}\right), \tag{A5}$$

where

$$s = \frac{k_s b_s^2 d^2}{4} \tag{A6}$$

is the warping spring constant at end a. Thus the warping spring stiffness is a function of k_s, b_s and d.

The values of k_s and b_s may be approximated indirectly from the judgement of the restraining effects of particular joint details. The warping restraint at a joint may be provided by an external medium such as a column to which the member is welded. Sometimes, column stiffeners are also provided [Fig. 1(b)] to achieve sufficient rigidity at the joint against distortion of flanges. In this case, the estimation of k_s and b_s may be dependent upon the local deformation of the column flange and column stiffeners present. Alternatively, warping restraint can be provided by an internal medium such as a pair of plate stiffeners welded to the flanges of the member itself [beam of Fig. 1(b)]. In that case, k_s and b_s may be approximated from the distance between the plate stiffeners and their in-plane shearing and flexural deformation (as expected from engineering judgement). In addition to the joint details, k_s and b_s may be dependent upon the type of member cross-section itself (Waldron, 1986).